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# PERIODIC SOLUTIONS OF STRONGLY QUADRATIC NON-LINEAR OSCILLATORS BY THE ELLIPTIC LINDSTEDT–POINCARÉ METHOD

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The elliptic Lindstedt–Poincaré method is used/employed to study the periodic solutions of quadratic strongly non-linear oscillators of the form $\ddot{x} + c_1x + c_2x^2 = \varepsilon f(x,. \dot{x})$, in which the Jacobian elliptic functions are employed instead of the usual circular functions in the classical Lindstedt–Poincaré method. The generalized Van de Pol equation with $f(x, \dot{x}) = \mu_0 + \mu_1 x - \mu_2 x^2$ is studied in detail. Comparisons are made with the solutions obtained by using the Lindstedt–Poincaré method and Runge–Kutta method to show the efficiency of the present method.



## 1. INTRODUCTION

Since 1969 when Barkham and Soudack [1] first used Jacobian elliptic functions to construct an approximate solution for the equation

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon f(x, \dot{x}, t), \tag{1.1}$$

many researchers have been developing various elliptic function methods such as the elliptic harmonic balance method, elliptic Krylov–Bogoliubov method, elliptic averaging method, elliptic Galerkin method, elliptic Rayleigh method, elliptic cubication technique and so on. This is well documented in the work of Yuste [2]. The authors have also presented two other elliptic function methods: elliptic perturbation method [3, 4] and elliptic Lindstedt–Poincaré method [5]. However, most of these methods are related to cubic non-linear oscillators, and very few of them have analyzed the equation with quadratic non-linearity. In this paper, the elliptic Lindstedt–Poincaré method [5] will be used to analyze the periodic solutions of quadratic non-linear oscillators of the form

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(x, \dot{x}) \tag{1.2}$$

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which are associated with many physical systems such as betatron oscillations and vibrations of shells. It is therefore also an important area of non-linear vibration investigation.

## 2. THE ELIPTIC LINDSTEDT–POINCARÉ METHOD

The elliptic Lindstedt–Poincaré (ELP) method was presented by the authors [5] for certain oscillators having cubic non-linearity. Now, we apply the ELP method for the equation having quadratic non-linearity

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon\, f(x, \dot{x}), \tag{2.1}$$

where $\varepsilon$ is a small parameter, and dots denote derivatives with respect to time $t$. We introduce a new variable $\tau$, and let

$$\tau = \omega t, \tag{2.2}$$

where $\omega$ is the non-linear frequency and will be determined later. Equation (2.1) then becomes

$$\omega^2\, x^n + c_1 x + c_2 x^2 = \varepsilon\, f(x, \omega x'), \tag{2.3}$$

in which primes denote derivatives with respect to the new variable $\tau$. Then let

$$x = \sum_{n=0}^{\infty} \varepsilon^n\, x_n(\tau), \qquad \omega = \sum_{n=0}^{\infty} \varepsilon^n\, \omega_n, \tag{2.4, 2.5}$$

where $\omega_n$ are constants and $x_n$ are assumed to be periodic functions. Substituting equations (2.4) and (2.5) into equation (2.3), expanding $f(x, \omega x')$ in power series of $\varepsilon$ and equating coefficients of like power of $\varepsilon$ yield the following equations:

$$\varepsilon^0 : \omega_0^2\, x_0'' + c_1 x_0 + c_2 x_0^2 = 0, \tag{2.6}$$

$$\varepsilon^1 : \omega_0^2\, x_1'' + (c_1 + 2c_2\, x_0) x_1 = f(x_0, \omega_0 x_0') - 2\omega_0 \omega_1\, x_0'' \tag{2.7}$$

$$\varepsilon^2 : \omega_0^2 x_2'' + (c_1 + 2c_2 x_0)\, x_2 = f_x'(x_0, \omega_0 x_0') x_1 + f_{\dot{x}}'(x_0, \omega_0 x_0')(\omega_0 x_1' + \omega_1 x_0') - (\omega_1^2 + 2\omega_0\omega_2)\, x_0'' - 2\omega_0\omega_1 x_1'' - c_2 x_1^2 \tag{2.8}$$

in which $f_x' = \partial f/\partial x$, $f_{\dot{x}}' \sim \partial f/\partial \dot{x}$. Equation (2.6) has an exact analytical solution which can be expressed by Jacobian elliptic functions in the case of $c_1 > 0$ and $c_2 > 0$:

$$x_0 = a_0\, \mathrm{cn}^2\, (\tau, k) + b_0, \qquad a_0 = 6\omega_0^2 k^2 / c_2, \tag{2.9, 2.10}$$

$$b_0 = -\,[4\omega_0^2\, (2k^2 - 1) + c_1]/2c_2, \qquad \omega_0^4 = c_1^2/[16(k^4 - k^2 + 1)], \tag{2.11, 2.12}$$

$\mathrm{cn}(\tau, k)$ is the cosine Jacobian elliptic function, $a_0$, $\omega_0$ and $k$ are called the amplitude, the angular frequency and the modulus of the elliptic function, respectively, and $b_0$ is called the bias. Obviously, the period of $x_0$ is $2K$, where $K$ is the complete elliptic integral of the first kind.

By multiplying both sides of equation (2.7) by $x'_0$ and then integrating the equation, one obtains

$$\omega_0^2 [x'_0 x'_1 - x''_0 x_1] \Big|_0^\tau + \int_0^\tau [\omega_0^2 x'''_0 + c_1 x'_0 + 2c_2 x_0 x'_0]\, x_1 \,\mathrm{d}\tau$$

$$= \int_0^\tau (f(x_0, \omega_0 x'_0) - 2\omega_0\omega_1 x''_0]\, x'_0 \,\mathrm{d}\tau. \tag{2.13}$$

Differentiating equation (2.6) with respect to $\tau$ leads to

$$\omega_0^2 x'''_0 + c_1 c'_0 + 2c_2 x_0 x'_0 = 0. \tag{2.14}$$

Note that $x_0$ is a periodic function with period $T = 2K$. Therefore, by letting $\tau = 2K$ in equation (2.13), one obtains

$$\int_0^{2K} [f(x_0, \omega_0 x'_0) - 2\omega_0\omega_1 x''_0]\, x'_0 \,\mathrm{d}\tau = 0. \tag{2.15}$$

Since

$$\int_0^{2K} x''_0 x'_0 \,\mathrm{d}\tau = \frac{1}{2} x'^2_0 \Big|_0^{2K} = 0, \tag{2.16}$$

$$\int_0^{2K} f(x_0, \omega_0 x'_0)\, x'_0 \,\mathrm{d}\tau = 0. \tag{2.17}$$

Therefore, the necessary condition for equation (2.1) to have a limit cycle is that the equation (2.17) has a non-zero solution. So $a_0$, $b_0$, $\omega_0$ and $k^2$ can be determined from equations (2.10)–(2.12) and (2.17).

It can been seen from equation (2.14) that $x'_0$ is a solution of the homogeneous part of equation (2.7). Therefore, the particular solution of equation (2.7) can be expressed by the following equation according to the theory of differential equations:

$$x_1 = x'_0 \int_0^\tau \frac{1}{x'^2_0} \left\{ \int_0^\tau \frac{x'_0}{\omega_0^2} [f(x_0, \omega_0 x'_0) - 2\omega_0\, \omega_1\, x''_0] \,\mathrm{d}\tau \right\} \mathrm{d}\tau. \tag{2.18}$$

Note that

$$x'_0 \int_0^\tau \frac{1}{x'^2_0} \left[ \int_0^\tau \frac{2\omega_1}{\omega_0} x'_0 x''_0 \,\mathrm{d}\tau \right] \mathrm{d}\tau = (\omega_1/\omega_0)\, x'_0\, \tau. \tag{2.19}$$

Hence equation (2.18) becomes

$$x_1 = x'_0 \int_0^\tau \frac{1}{x'^2_0} \left\{ \int_0^\tau \frac{x'_0}{\omega_0^2} f(x_0, \omega_0 x'_0) \,\mathrm{d}\tau \right\} \mathrm{d}\tau - (\omega_1/\omega_0)\, x'_0 \tau. \tag{2.20}$$

The term $(\omega_1/\omega_0) x'_0 \tau$ is called a secular term. It tends to infinity as $\tau \to \pm\infty$. However, $x_1/x_0$ should be bounded for all $\tau$. If $f(x_0, \omega_0\, x'_0)$ does not contain the term $x''_0$ explicitly or implicitly, then $\omega_1$ must vanish, i.e.,

$$\omega_1 = 0. \tag{2.21}$$

One can continue the perturbation procedure to determinate the next order solution $x_2$ and $\omega_2$.

It is worth pointing out that when $c_1 > 0$, $c_2 < 0$, the solution of equation (2.6) can be expressed by

$$x_0(\tau) = \bar{a}_0 \operatorname{sn}^2 \tau + \bar{b}_0, \tag{2.22}$$

where

$$\bar{a}_0 = -a_0, \qquad \bar{b}_0 = a_0 + b_0. \tag{2.23, 2.24}$$

It can be shown that equation (2.22) is indeed identical to equation (2.9), because

$$a_0 \operatorname{cn}^2 \tau + b_0 = a_0 (1 - \operatorname{sn}^2 \tau) + b_0 = \bar{a}_0 \operatorname{sn}^2 \tau + \bar{b}_0.$$

Similarly, when $c_1 < 0$, $c_2 > 0$, the solution of equation (2.6) can be expressed by

$$x_0(\tau) = \bar{\bar{a}}_0 \operatorname{dn}^2 \tau + \bar{\bar{b}}_0. \tag{2.25}$$

Here

$$\bar{\bar{a}}_0 = a_0/k^2, \qquad \bar{\bar{b}}_0 = b_0 - a_0 (1 - k^2)/k^2. \tag{2.26, 2.27}$$

It can also be proved that equation (2.25) is equivalent to equation (2.9). Therefore, one can use equations (2.9)–(2.12) as a unified solution of equation (2.6) later.

## 3. THE GENERALIZED VAN DER POL OSCILLATOR

As an application of the ELP method, we consider the generalized Van der Pol oscillator

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon(\mu_0 + \mu_1 x - \mu_2 x^2)\, \dot{x}. \tag{3.1}$$

Here $f(x, \dot{x}) = (\mu_0 + \mu_1 x - \mu_2 x^2)\, \dot{x}$. Let

$$I_1(\tau) = \int_0^\tau f(x_0, \omega_0 x_0')\, x_0'\, d\tau. \tag{3.2}$$

Substituting equation (2.9) into (3.2), one obtains

$$I_1(\tau) = 4\omega_0 a_0^2 \left[C_a I_{11}(\tau) + C_b I_{12}(\tau) + C_c I_{13}(\tau)\right], \tag{3.3}$$

where

$$C_a = \mu_0 + \mu_1 b_0 - \mu_2 b_0^2, \qquad C_b = \mu_1 a_0 - 2\mu_2 a_0 b_0, \qquad C_c = -\mu_2 a_0^2,$$

$$\begin{aligned} I_{11}(\tau) = \int_0^\tau & \operatorname{sn}^2 \tau \operatorname{cn}^2 \tau \operatorname{dn}^2 \tau \, d\tau \\ = \{ & [(1 - k^2)(k^2 - 2) + 2(k^4 - k^2 + 1)\, E/K]\tau \\ & + 2(k^4 - k^2 + 1) Z(\tau) - 3k^4 \operatorname{sn} \tau \operatorname{cn}^3 \tau \operatorname{dn} \tau \\ & + k^2 (2k^2 - 1) \operatorname{sn} \tau \operatorname{cn} \tau \operatorname{dn} \tau \}/(15k^4), \end{aligned}$$

$$
\begin{aligned}
I_{12}(\tau) &= \int_0^\tau \mathrm{sn}^2\,\tau\,\mathrm{cn}^4\,\tau\ \mathrm{dn}^2\ \tau\ \mathrm{d}\tau \\
&= \{[(1-k^2)(3k^4-15k^2+8)+(2k^2-1)\,(3k^4-3k^2+8)E/K]\tau \\
&\quad + (2k^2-1)(3k^4-3k^2+8)Z(\tau)+k^2[4(2k^2-1)^2 \\
&\quad + 10k^2(1-k^2)]\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau \\
&\quad + 3k^4(2k^2-1)\,\mathrm{sn}\,\tau\,\mathrm{cn}^3\,\tau\,\mathrm{dn}\,\tau - 15k^6\,\mathrm{sn}\,\tau\,\mathrm{cn}^5\,\tau\,\mathrm{dn}\,\tau\}/(105k^6)
\end{aligned}
$$

$$
\begin{aligned}
I_{13}(\tau) &= \int_0^\tau \mathrm{sn}^2\,\tau\,\mathrm{cn}^6\,\tau\,\mathrm{dn}^2\,\tau\,\mathrm{d}\tau \\
&= \{[(1-k^2)(5k^6-45k^4+48k^2-16)+(10k^8-20k^6+66k^4-56k^2+16)E/K]\,\tau \\
&\quad + (10k^8-20k^6+66k^4-56k^2+16)Z(\tau) \\
&\quad + k^2\,(2k^2-1)\,[8(2k^2-1)^2+27k^2\,(1-k^2)]\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{d}n\,\tau \\
&\quad + k^4\,[6(2k^2-1)^2+14k^2(1-k^2)]\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau \\
&\quad + 5k^6\,(2k^2-1)\,\mathrm{sn}\,\tau\,\mathrm{cn}^5\,\tau\,\mathrm{dn}\,\tau - 35k^8\,\mathrm{sn}\,\tau\,\mathrm{cn}^7\,\tau\,\mathrm{dn}\,\tau\}/(315k^8).
\end{aligned}
$$

Hence equation (3.3) can be further expressed as

$$
\begin{aligned}
I_1(\tau) &= 4\omega_0 a_0^2\,[(C_a I_{11}^k + C_b I_{12}^k + C_c I_{13}^k)\tau + C_{11}\,Z(\tau) + C_{12}\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau \\
&\quad + C_{13}\,\mathrm{sn}\,\tau\,\mathrm{Cn}^3\,\tau\,\mathrm{dn}\,\tau + C_{14}\,\mathrm{sn}\,\tau\,\mathrm{cn}^5\,\tau\,\mathrm{dn}\,\tau + C_{15}\,\mathrm{dn}\,\tau\,\mathrm{cn}^7\,\tau\,\mathrm{dn}\,\tau. \qquad (3.4)
\end{aligned}
$$

Using condition of equation (2.17) and the periodic property of elliptic functions, one has

$$
C_a I_{11}^k + C_b I_{12}^k + C_c I_{13}^k = 0, \qquad (3.5)
$$

where

$$
\begin{aligned}
I_{11}^k &= [(1-k^2)\,(k^2-2)+2\,(k^4-k^2+1)E/K]/(15k^4), \\
I_{12}^k &= [(1-k^2)\,(3k^2-15k^2+8)+(2k^2-1)(3k^4-3k^2+8)E/K]/(105k^6), \\
I_{13}^k &= [(1-k^2)\,(5k^6-45k^4+48k^2-16) \\
&\quad + (10k^8-20k^6+66k^4-56k^2+16)E/K]/(315k^8).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_1(\tau) &= 4\omega_0 a_0^2\,[C_{11}\,Z(\tau) + C_{12}\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau\,\mathrm{dn}\,\tau + C_{13}\,\mathrm{sn}\,\tau\,\mathrm{cn}^3\,\tau\,\mathrm{dn}\,\tau \\
&\quad + C_{14}\,\mathrm{sn}\,\tau\,\mathrm{cn}^5\,\tau\,\mathrm{dn}\,\tau + C_{15}\,\mathrm{sn}\,\tau\,\mathrm{cn}^7\,\tau\,\mathrm{dn}\,\tau]. \qquad (3.6)
\end{aligned}
$$

$Z(\tau)$ is called Jacobi Zeta function with period $2K$, and is defined [6] by

$$
Z(\tau) = E(\tau) - \frac{E}{K}\,\tau, \qquad (3.7)
$$

where $E(\tau)$ is the elliptic integral of the second kind and $E$ is the complete integral of $E(\tau)$. Note that $f(x_0,\ \omega_0 x_0') = (\mu_0 + \mu_1 x_0 - \mu_2 x_0^2)\,\omega_0\,x_0'$; obviously, it does not contain the term $x_0''$, so $\omega_1 = 0$ and equation (2.18) becomes

$$x_1 = x_0' \int_0^\tau \frac{1}{\omega_0^2 x_0'^2} I_1(\tau)\, \mathrm{d}\tau, \tag{3.8}$$

Substituting equation (3.6) into equation (3.8) and integrating it, one finally obtains $x_1$ as

$$x_1 = (x_0'/\omega_0) \sum_{j=1}^{5} C_{1j} D_{1j} + C_{10} x_0', \tag{3.9}$$

$$x_1' = (x_0''/\omega_0) \sum_{j=1}^{5} C_{1j} D_{1j} + (x_0'/\omega_0) \sum_{j=1}^{5} C_{1j} S_{1j} + C_{10} x_0'' \tag{3.10}$$

in which

$$D_{1j} = \int S_{1j}\, \mathrm{d}\tau \quad (j = 1, 2, \dots, 5), \qquad C_{10} = (-1/\omega_0) \lim_{\tau \to 0} C_{1j} D_{1j}.$$

The coefficients of $C_{1j}$, $D_{1j}$ and $S_{1j}$ are listed in Appendix A.

## 4. EXAMPLES

**Example 1.** Consider the equation

$$\ddot{x} + 8x - x^2 = \varepsilon\,(1 - x^2)\,\dot{x}. \tag{4.1}$$

In this example, $c_1 = 8$, $c_2 = -1$, $\mu_0 = 1$, $\mu_1 = 0$ and $\mu_2 = 1$. One gets $k = 0{\cdot}5429$ from equation (3.5) and $a_0 = -3{\cdot}974$, $b_0 = 2{\cdot}155$, $\omega_0 = 1{\cdot}499$ from equations (2.10) to (2.12). Then one gets the coefficients $C_{1j}$ from formulas listed in Appendix A. $C_{10} = 6{\cdot}5310$, $C_{11} = -55{\cdot}5437$, $C_{12} = 11{\cdot}1035$, $C_{13} = -2{\cdot}2156$, $C_{14} = -2{\cdot}0980$, $C_{15} = 1{\cdot}7549$. The solution to $O(\varepsilon^2)$ is $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, where $x_0$ and $x_1$ are taken from equation (2.9) and equation (3.11) respectively. One also obtains the solution of the classical L–P method from Appendix B:

$$x = 0{\cdot}25 + 2\cos\tau - 0{\cdot}0833\cos 2\tau + \varepsilon(0{\cdot}2652\sin\tau - 0{\cdot}0839\sin 3\tau).$$

The limit cycle phase portraits for the cases $\varepsilon = 0{\cdot}1$ and $\varepsilon = 0{\cdot}3$ are shown in Figure 1. Comparisons are also made with the results of the numerical integration method (in examples of this paper, the fourth order Runge–Kutta (R–K) method is employed) and the classical Lindstedt–Poincaré (L–P) method. It can be seen from Figure 1 that the solutions obtained by the present method are very close to those obtained by the fourth order R–K method for the both cases of $\varepsilon = 0{\cdot}1$ and $\varepsilon = 0{\cdot}3$, while the solution of the L–P method has obvious errors when $\varepsilon = 0{\cdot}3$.

**Example 2.** Consider the equation

$$\ddot{x} + x + x^2 = \varepsilon(0{\cdot}1 + x - x^2)\,\dot{x}. \tag{4.2}$$

In this example, $c_1 = 1$, $c_2 = 1$, $\mu_0 = 0{\cdot}1$, $\mu_1 = 1$ and $\mu_2 = 1$. One gets $k = 0{\cdot}7114$ from equation (3.5) and $a_0 = 0{\cdot}8766$, $b_0 = -0{\cdot}5071$, $\omega_0 = 0{\cdot}5373$ from equations

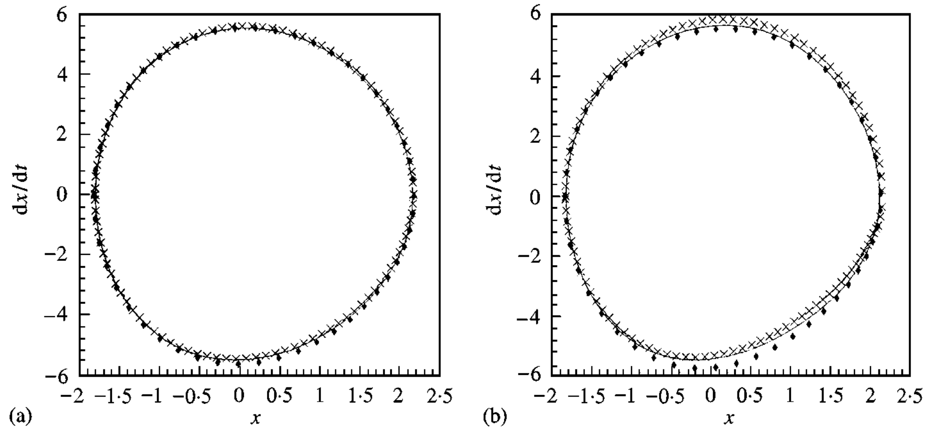


Figure 1. Limit cycles of equation (4.1). (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}3$; (——) R–K method; ( + ) present method; (◆) L–P method.

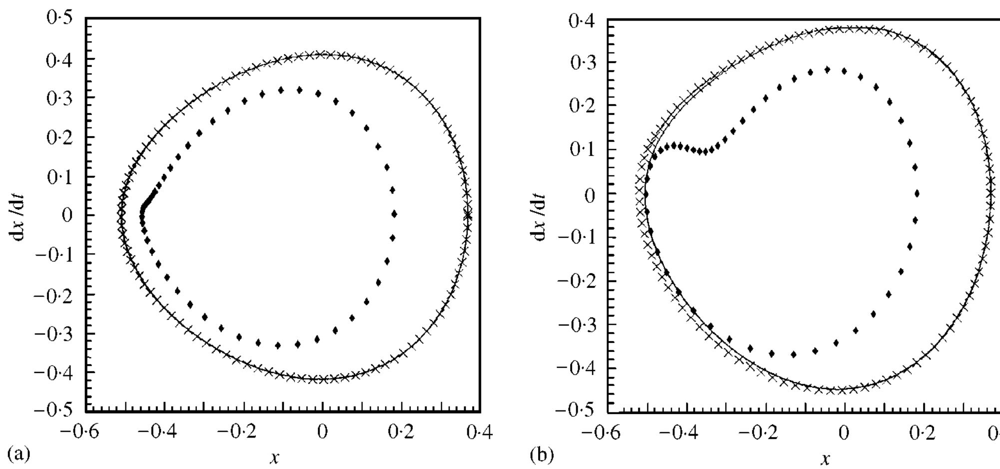


Figure 2. Limit cycles of equation (4.2). (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}5$; (——) R–K method; ( + ) present method; (◆) L–P method.

(2.10) to (2.12). Then one gets the coefficients $C_{1j}$ as follows: $C_{10} = 0{\cdot}6707$, $C_{11} = -0{\cdot}3454$, $C_{12} = 0{\cdot}1615$, $C_{13} = 0{\cdot}1007$, $C_{14} = -0{\cdot}2525$, $C_{15} = -0{\cdot}0854$. One also obtains the solution of the classical L–P method from Appendix B:

$$x = -0{\cdot}2 + 0{\cdot}3163\cos\tau + 0{\cdot}0667\cos 2\tau$$

$$+\,\varepsilon(-0{\cdot}1096\sin\tau + 0{\cdot}0667\sin 2\tau - 0{\cdot}0079\sin 3\tau).$$

The limit cycle phase portriats for the cases $\varepsilon = 0{\cdot}1$ and $\varepsilon = 0{\cdot}5$ are shown in Figure 2. Comparisons are also made with the results of the numerical intergration method and the classical L–P method. It can be seen from Figure 2 that the solutions obtained by the present method are nearly identical with those obtained by the sourth order R–K method for both cases of $\varepsilon = 0{\cdot}1$ and $\varepsilon = 0{\cdot}5$, while the

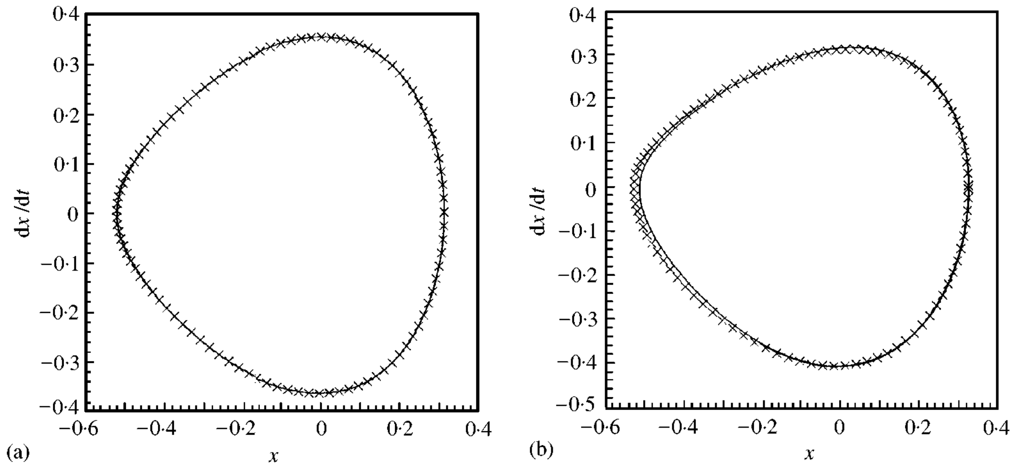


Figure 3. Limit cycles of equation (4.3). (a) $\varepsilon = 0{\cdot}1$; (b) $\varepsilon = 0{\cdot}8$; (——) R–K method; ( + ) present method.

departure of the solutions of the L–P method is very large for both cases. It can be said that the results are not acceptable at all.

**Example 3.** Consider the equation

$$\ddot{x} + x + 1{\cdot}5x^2 = \varepsilon(0{\cdot}07 + x)\,\dot{x}. \tag{4.3}$$

In this example, $c_1 = 1$, $c_2 = 1{\cdot}5$, $\mu_0 = 0{\cdot}07$, $\mu_1 = 1$ and $\mu_2 = 0$. One gets $k = 0{\cdot}8667$ from equation (3·5) and $a_0 = 0{\cdot}8330$, $b_0 = -0{\cdot}5190$, $\omega_0 = 0{\cdot}5265$ from equations (2.10) to (2.12). Then one gets the coefficient $C_{1j}$ as follows: $C_{10} = 0{\cdot}3156$, $C_{11} = -0{\cdot}0163$, $C_{12} = -0{\cdot}0205$, $C_{13} = 0{\cdot}1057$, $C_{14} = -0{\cdot}1190$, $C_{15} = 0{\cdot}0$. The limit cycle phase portriats for the cases $\varepsilon = 0{\cdot}1$ and 0·8 are shown in Figure 3. Comparisons are also made with the results of the numerical integration method. It can be seen from Figure 3 that the solutions obtained by the present method are nearly identical with those given by the fourth order R–K method for both cases of $\varepsilon = 0{\cdot}1$ and $\varepsilon = 0{\cdot}8$. However, by using the classicial L–P method one obtains $x = 0$. Because $\mu_2 = 0$ in this example, it turns out that $M_0 = 0$ and therefore the solution is trivial.

## 5. CONCLUSION

The elliptic Lindstedt–Poincaré (ELP), method is an efficient method for calculating periodic solutions of strongly quadratic non-linear oscillators especially for those equations such as $\mu_2 = 0$ or $c_2 > 0$, $\mu_1/\mu_0 > 0$ in which the classical L–P method cannot be used. All the examples show that the results of the present method are in excellent agreement with those obtained by the numerical integration method even for moderately large values of the parameter $\varepsilon$.

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## APPENDIX A

The coefficients $C_{1j}$, $D_{1j}$ and $S_{1j}$ occurring in equation (3.9) are

$$C_{11} = 2C_a(k^4 - k^2 + 1)/(15k^4) + C_b(2k^2 - 1)(3k^4 - 3k^2 + 8)/(105k^6)$$
$$+ C_c(10k^8 - 20k^6 + 66k^4 - 56k^2 + 16)/(315k^8),$$

$$C_{12} = C_a(2k^2 - 1)/(15k^2) + C_b[4(2k^2 - 1)^2 + 10k^2(1 - k^2)]/(105k^4)$$
$$+ C_c(2k^2 - 1)[8(2k^2 - 1)^2 + 27k^2(1 - k^2)]/(315\,k^6),$$

$$C_{13} = -C_a/5 + C_b(2k^2 - 1)/(35k^2) + C_c[6(2k^2 - 1)^2 + 14k^2(1 - k^2)]/(315\,k^4),$$

$$C_{14} = -C_b/7 + C_c(2k^2 - 1)/(63k^2),$$

$$C_{15} = -C_c/9,$$

$$S_{11} = Z(\tau)/(\mathrm{sn}^2\tau\,\mathrm{cn}^2\tau\,\mathrm{dn}^2\tau), \quad S_{12} = 1/(\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau); \quad S_{13} = \mathrm{cn}\,\tau/(\mathrm{sn}\,\tau\,\mathrm{dn}\,\tau),$$

$$S_{14} = \mathrm{cn}^3\,\tau/(\mathrm{sn}\,\tau\,\mathrm{dn}\,\tau), \qquad S_{15} = \mathrm{cn}^5\,\tau/(\mathrm{sn}\,\tau\,\mathrm{dn}\,\tau),$$

$$D_{11} = (H_1 + H_2 + H_3 + H_4 + H_5)/k'^4,$$

$$H_1 = (\mathrm{sn}\,\tau\,\mathrm{dn}\,\tau/\mathrm{cn}\,\tau - k'^4\,\mathrm{cn}\,\tau\,\mathrm{dn}\,\tau/\mathrm{sn}\,\tau + k^6\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau/\mathrm{dn}\,\tau)\,Z(\tau),$$

$$H_2 = [k'^2(1 + k'^2) - (1 + k^4 + k'^4)\,E/K]\,\ln\,\Theta(\tau),$$

$$H_3 = -(1 + k^4 + k'^4)Z^2(\tau)/2,$$

$$H_4 = k'^2\,\ln(\mathrm{cn}\,\tau) + k'^4\ln(\mathrm{sn}\,\tau) + (k^2/2)\mathrm{cn}^2\,\tau - (k^2k'^4/2)\,\mathrm{sn}^2\,\tau + (k^4/2)\,\mathrm{dn}^2\,\tau,$$

$$H_5 = (E/K)\,[-\ln(\mathrm{cn}\,\tau) - k'^4\,\ln(\mathrm{sn}\,\tau) - k^4\ln(\mathrm{dn}\,\tau)],$$

where

$k'^2 = 1 - k^2$,

$\ln \Theta(\tau) = \ln\ \xi - 2 \sum \dfrac{q^m}{m(1 - q^{2m})} \cos(2mv)$,

$\xi = 0{\cdot}57721566, \qquad q = \mathrm{e}^{-(\pi K'/K)}, \qquad v = \pi\tau/2K$,

$D_{12} = \ln(\mathrm{sn}\,\tau) - \ln(\mathrm{cn}\,\tau)/k'^2 + (k^2/k'^2)\ln(\mathrm{dn}\ \tau)$,

$D_{13} = \ln(\mathrm{sn}\,\tau/\mathrm{dn}\ \tau)$,

$D_{14} = \ln(\mathrm{sn}\,\tau) + (k'^2/k^2)\ \ln\ (\mathrm{dn}\,\tau)$,

$D_{15} = \ln(\mathrm{sn}\,\tau) + cn^2\,\tau/2k^2 - (k'^4/k^4)\ln(\mathrm{dn}\ \tau)$.

## APPENDIX B

The solution of equation (3.1) by using the Lindstedt–Poincaré method:

Let

$$\tau = \omega t \quad \text{and} \quad \bar{c}_2 = c_2/\varepsilon.$$

Then equation (3.1) becomes

$$\omega^2 x'' + c_1 x + \varepsilon\,\bar{c}_2\, x^2 = \varepsilon(\mu_0 + \mu_1 x - \mu_2\, x^2)\,\omega x'.$$

Using the classicial L–P procedure, one finally obtains

$$\omega = \omega_0 + \varepsilon^2\omega_2 + O(\varepsilon^3), \qquad x = x_0 + \varepsilon x_1 + O(\varepsilon^2),$$

where

$\omega_0 = \sqrt{c_1}$,

$\omega_2 = (64\ \mu_0\,\mu_1\,M_0 - 16\mu_1^2 M_0^2 - 18\mu_0\mu_2 M_0^2 - 16\mu_1\mu_2 M_0^3 + 3\mu_2^2 M_0^4)/(384\omega_0)$
$\qquad - 5\bar{c}_2^3\ M_0^2/(12\omega_0^3)$,

$x_0 = M_0 \cos\tau$,

$x_1 = C_{c0} + M_1 \cos\tau + N_1 \sin\tau + C_{c2}\cos 2\tau + C_{s2}\sin 2\tau + C_{s3}\sin 3\tau$,

$M_0 = 2\sqrt{\mu_0/\mu_2}$ (if $\mu_2 = 0$, then $M_0 = 0$), $\quad M_1 = -\mu_1\bar{c}_2 M_0^3/(8\omega_0^2\,\mu_0)$,

$N_1 = 3\ \mu_2 M_0^3/(32\omega_0) - \mu_1\,M_0^2/(3\omega_0)$,

$C_{c0} = -\bar{c}_2\,M_0^2/(2\omega_0^2), \qquad C_{c2} = \bar{c}_2\,M_0^2/(6\omega_0^2)$,

$C_{s2} = \mu_1\,M_0^2/(6\omega_0), \qquad C_{s3} = -\mu_2\,M_0^3/(32\omega_0)$.